Thus Eq. (2.43) asserts that if we have successive mappings $\mathbf{u} = \mathbf{g}(\mathbf{x})$, $\mathbf{y} = \mathbf{f}(\mathbf{u})$ and hence obtain a composite mapping $\mathbf{y} = \mathbf{f}(\mathbf{g}(\mathbf{x}))$, then the matrix of the linear approximation of the composite mapping is obtained by *multiplying* the approximating matrices of the two stages. In the special case when $\mathbf{f}$ and $\mathbf{g}$ are linear, then we have $\mathbf{u} = B\mathbf{x}$, $\mathbf{y} = A\mathbf{u}$ for appropriate matrices A and B ($\mathbf{u_x} = B$, $\mathbf{y_u} = A$), and the composite mapping is $\mathbf{y} = A(B\mathbf{x}) = AB\mathbf{x}$, as in Section 1.8; thus $\mathbf{y_x} = AB = \mathbf{y_u}\mathbf{u_x}$.

Case of square matrices. In the preceding analysis, let $m = n = p$, so that all the Jacobian matrices appearing are *square* and each has a determinant—the *Jacobian determinant* of the corresponding mapping, as in Section 2.7. For example,

$$\det \mathbf{y_u} = \begin{vmatrix} \frac{\partial y_1}{\partial u_1} & \cdots & \frac{\partial y_1}{\partial u_n} \\ \vdots & & \vdots \\ \frac{\partial y_n}{\partial u_1} & \cdots & \frac{\partial y_n}{\partial u_n} \end{vmatrix} = \frac{\partial(y_1, \dots, y_n)}{\partial(u_1, \dots, u_n)}.$$

To the equation (2.49) we can apply the rule $\det AB = \det A \det B$ (Eq. (1.60) in Section 1.9) to obtain the following very useful rule:

$$\det \mathbf{y_x} = \det \mathbf{y_u} \det \mathbf{u_x}; \qquad \textbf{(2.50)}$$

that is,

$$\frac{\partial(y_1, \dots, y_n)}{\partial(x_1, \dots, x_n)} = \frac{\partial(y_1, \dots, y_n)}{\partial(u_1, \dots, u_n)} \frac{\partial(u_1, \dots, u_n)}{\partial(x_1, \dots, x_n)}. \qquad \textbf{(2.51)}$$

If, for example, $n = 2$, then each determinant here can be interpreted as in Section 2.7 as plus or minus the ratio of small corresponding areas, and (2.51) states roughly that

$$\frac{\Delta A_y}{\Delta A_x} = \frac{\Delta A_y}{\Delta A_u} \frac{\Delta A_u}{\Delta A_x},$$

where we have written ΔA_x for an "area element" in the x_1x_2-plane, and similarly for ΔA_y, ΔA_u. There is a similar interpretation for $n = 3$, in terms of volumes, and for higher n in terms of higher-dimensional volume.

PROBLEMS

1. Find the Jacobian matrix $(\partial y_i/\partial x_j)$ in the form of a product of two matrices and evaluate the matrix for the given values of $x_1, x_2, \dots$.

a) $y_1 = u_1u_2 - 3u_1$, $y_2 = u_2^2 + 2u_1u_2 + 2u_1 - u_2$; $u_1 = x_1 \cos 3x_2$, $u_2 = x_1 \sin 3x_2$; $x_1 = 0$, $x_2 = 0$.

b) $y_1 = u_1^2 + u_2^2 - 3u_1 + u_3$, $y_2 = u_1^2 - u_2^2 + 2u_1 - 3u_3$; $u_1 = x_1x_2x_3^2$, $u_2 = x_1x_2^2x_3$, $u_3 = x_1^2x_2x_3$; $x_1 = 1$, $x_2 = 1$, $x_3 = 1$.

c) $y_1 = u_1e^{u_2}$, $y_2 = u_1e^{-u_2}$, $y_3 = u_1^2$; $u_1 = x_1^2 + x_2$, $u_2 = 2x_1^2 - x_2$; $x_1 = 1$, $x_2 = 0$.

d) $y_1 = u_1^2 + \cdots + u_n^2 - u_1^2$, $y_2 = u_1^2 + \cdots + u_n^2 - u_2^2, \dots, y_n = u_1^2 + \cdots + u_n^2 - u_n^2$; $u_1 = x_1^2 + x_1x_2$, $u_2 = x_1^2 + 2x_1x_2, \dots, u_n = x_1^2 + nx_1x_2$; $x_1 = 1$, $x_2 = 0$.

2. a) Find $\partial(z, w)/\partial(x, y)$ for $x = 1$, $y = 0$ if $z = u^3 + 3u^2v - v^3 + u^2 - v^2$, $w = u^3 + v^3 - 2u^2$; $u = x\cos xy$, $v = x\sin xy + x^2 - y^2$.

b) Find $\partial(x, y)/\partial(s, t)$ for $s = 0$, $t = 0$ if $x = (z^2 + w^2)^{1/2}$, $y = w(z^2 + w^2)^{-1/2}$; $z = (s + t + 1)^{-1}$, $w = (2s - t + 1)^{-1}$.

3. Justify the rules, under appropriate hypotheses:

a) If $\mathbf{y} = \mathbf{f}(\mathbf{u})$, $\mathbf{u} = \mathbf{g}(\mathbf{v})$, $\mathbf{v} = \mathbf{h}(\mathbf{x})$, then $\mathbf{y_x} = \mathbf{y_u u_v v_x}$.

b) $\dfrac{\partial(z, w)}{\partial(x, y)} = \dfrac{\partial(z, w)}{\partial(u, v)}\dfrac{\partial(u, v)}{\partial(s, t)}\dfrac{\partial(s, t)}{\partial(x, y)}$.

4. For certain functions $f(x, y)$, $g(x, y)$, $p(u, v)$, $q(u, v)$ it is known that $f(x_0, y_0) = u_0$, $g(x_0, y_0) = v_0$ and that $f_x(x_0, y_0) = 2$, $f_y(x_0, y_0) = 3$, $g_x(x_0, y_0) = -1$, $g_y(x_0, y_0) = 5$, $p_u(u_0, v_0) = 7$, $p_v(u_0, v_0) = 1$, $q_u(u_0, v_0) = -3$, $q_v(u_0, v_0) = 2$. Let $z = F(x, y) = p(f(x, y), g(x, y))$, $w = G(x, y) = q(f(x, y), g(x, y))$ and find the Jacobian matrix of $z(x, y)$, $w(x, y)$ at (x_0, y_0).

5. Let $u_1 = x_1 - 3x_2 + 2x_1x_2$, $u_2 = 2x_1 + 5x_2 - 3x_1x_2$. Let $\mathbf{w} = (w_1, w_2)$ be a vector function of $\mathbf{u} = (u_1, u_2)$ such that $\mathbf{w_u} = \begin{bmatrix} 2 & 11 \\ 7 & 5 \end{bmatrix}$ for $\mathbf{u} = (3, 3)$. Find the Jacobian matrix at $\mathbf{x} = (2, 1)$ for the composite function $\mathbf{w}[\mathbf{u}(\mathbf{x})]$.

6. Let $\mathbf{u} = \mathbf{f}(\mathbf{x})$ and $\mathbf{v} = \mathbf{g}(\mathbf{x})$ be differentiable mappings from a domain D in 3-dimensional space to 3-dimensional space. Let a and b be constant scalars. Let A be a constant 3×3 matrix. Show:

a) $d(\mathbf{u} + \mathbf{v}) = d\mathbf{u} + d\mathbf{v}$

b) $d(a\mathbf{u} + b\mathbf{v}) = a\,d\mathbf{u} + b\,d\mathbf{v}$

c) $d(A\mathbf{u}) = A\,d\mathbf{u}$

d) $d(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot d\mathbf{v} + \mathbf{v} \cdot d\mathbf{u}$

e) $d(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times d\mathbf{v} + d\mathbf{u} \times \mathbf{v}$

2.10 Implicit Functions

If $F(x, y, z)$ is a given function of x, y, and z, then the equation

$$F(x, y, z) = 0 \tag{2.52}$$

is a relation that may describe one or several functions z of x and y. Thus if $x^2 + y^2 + z^2 - 1 = 0$, then

$$z = \sqrt{1 - x^2 - y^2} \quad \text{or} \quad z = -\sqrt{1 - x^2 - y^2},$$

both functions being defined for $x^2 + y^2 \leq 1$. Either function is said to be *implicitly defined* by the equation $x^2 + y^2 + z^2 - 1 = 0$.

Similarly, an equation

$$F(x, y, z, w) = 0 \tag{2.53}$$

may define one or more implicit functions w of x, y, z. If two such equations are given:

$$F(x, y, z, w) = 0, \qquad G(x, y, z, w) = 0, \tag{2.54}$$