PROBLEMS

1. If (a) $y = u + v$, (b) $y = u \cdot v$, (c) $y = u/v$, where u and v are functions of x, then apply (2.33) to find dy/dx.
2. If $y = u^v$, where u and v are functions of x, then find dy/dx by (2.33). [Hint: $(a^x)' = a^x \log a$, $(x^a)' = ax^{a-1}$.]
3. If $y = \log_u v$, where u and v are functions of x, then find dy/dx by (2.33). [Hint: $(\log_a x)' = 1/[x \log a]$, $\log_x a = 1/\log_a x$.]
4. If $z = e^x \cos y$, while x and y are implicit functions of t defined by the equations
$$x^3 + e^x - t^2 - t = 1, \qquad yt^2 + y^2 t - t + y = 0,$$
then find dz/dt for $t = 0$. [Note that $x = 0$ and $y = 0$ for $t = 0$.]
5. Let $z = x^3 - 3x^2 y$, where x and y are functions of t such that for $t = 5$, $x = 7$, $y = 2$, $dx/dt = 3$, and $dy/dt = -1$. Find dz/dt for $t = 3$.
6. Let $z = f(x, y)$, where $f_x(4, 4) = 7$, $f_y(4, 4) = 9$, $x = 2e^{3t} + t^2 - t + 2$, $y = 5e^{3t} + 3t - 1$. Find dz/dt for $t = 0$.
7. If $u = f(x, y)$ and $x = r\cos\theta$, $y = r\sin\theta$, then show that
$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2.$$
[Hint: Use the chain rules to evaluate the derivatives on the *right*-hand side.]
8. If $w = f(x, y)$ and $x = u\cosh v$, $y = u\sinh v$, then show that
$$\left(\frac{\partial w}{\partial x}\right)^2 - \left(\frac{\partial w}{\partial y}\right)^2 = \left(\frac{\partial w}{\partial u}\right)^2 - \frac{1}{u^2}\left(\frac{\partial w}{\partial v}\right)^2.$$
[Cf. hint for Problem 7.]
9. If $z = f(ax + by)$, show that
$$b\frac{\partial z}{\partial x} - a\frac{\partial z}{\partial y} = 0.$$
10. Find $\partial z/\partial x$ and $\partial z/\partial y$ by first obtaining dz:
   a) $z = \log\sin(x^2y^2 - 1)$
   b) $z = x^2y^2\sqrt{1 - x^2 - y^2}$
   c) $x^2 + 2y^2 - z^2 = 1$
11. If $f(x, y)$ satisfies the identity
$$f(tx, ty) = t^n f(x, y)$$
for a fixed n, f is called *homogeneous* of degree n. Show that one then has the relation
$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y).$$
This is *Euler's theorem on homogeneous functions*. [Hint: Differentiate both sides of the identity with respect to t and then set $t = 1$.]
12. (*The Stokes total time derivative in hydrodynamics*) Let $w = F(x, y, z, t)$, where $x = f(t)$, $y = g(t)$, $z = h(t)$, so that w can be expressed in terms of t alone.

Show that

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} + \frac{\partial w}{\partial t}.$$

Here both dw/dt and $\partial w/\partial t = F_t(x, y, z, t)$ have meaning and are in general unequal. In hydrodynamics, dx/dt, dy/dt, dz/dt are the velocity components of a moving fluid particle, and dw/dt describes the variation of w "following the motion of the fluid." It is customary, following Stokes, to write Dw/Dt for dw/dt. [See H. Lamb, *Hydrodynamics*, 6th ed. (Dover: New York, 1945), p. 3.]

2.9 The General Chain Rule

On occasion, one deals with two sets of functions:

$$\begin{aligned} y_1 &= f_1(u_1, \ldots, u_p), \\ &\vdots \\ y_m &= f_m(u_1, \ldots, u_p), \end{aligned} \tag{2.37}$$

and

$$\begin{aligned} u_1 &= g_1(x_1, \ldots, x_n), \\ &\vdots \\ u_p &= g_p(x_1, \ldots, x_n). \end{aligned} \tag{2.38}$$

If one substitutes the functions (2.38) in the functions (2.37), one obtains composite functions

$$\begin{aligned} y_1 &= f_1(g_1(x_1, \ldots, x_n), \ldots, g_p(x_1, \ldots, x_n)) = F_1(x_1, \ldots, x_n), \\ &\vdots \\ y_m &= f_m(g_1(x_1, \ldots, x_n), \ldots, g_p(x_1, \ldots, x_n)) = F_m(x_1, \ldots, x_n). \end{aligned} \tag{2.39}$$

Under the appropriate hypotheses, one can obtain the partial derivatives of these composite functions by chain rules, as in the previous section:

$$\frac{\partial y_i}{\partial x_j} = \frac{\partial y_i}{\partial u_1}\frac{\partial u_1}{\partial x_j} + \cdots + \frac{\partial y_i}{\partial u_p}\frac{\partial u_p}{\partial x_j} \quad (i = 1, \ldots, m,\ j = 1, \ldots, n). \tag{2.40}$$

The formulas (2.40) can be expressed concisely in matrix language. The partial derivatives $\partial y_i/\partial x_j$ are the entries in the $m \times n$ matrix

$$\left(\frac{\partial y_i}{\partial x_j}\right) = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix}. \tag{2.41}$$