ratio of corresponding *lengths* Δy, Δx: $|dy/dx| \sim |\Delta y|/|\Delta x|$ for small Δx, since $dy/dx = \lim(\Delta y/\Delta x)$ as $\Delta x \to 0$.

The Jacobian determinant is also denoted as follows:

$$J = \frac{\partial(y_1, \ldots, y_n)}{\partial(x_1, \ldots, x_n)} = \frac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)}.$$

The concept of Jacobian determinant and these notations can also be applied to n functions of more than n variables. One simply forms the indicated partial derivatives, holding all other variables constant. For example, for $f(u, v, w)$, $g(u, v, w)$, one has

$$\frac{\partial(f, g)}{\partial(u, v)} = \begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix},$$

$$\frac{\partial(f, g)}{\partial(u, w)} = \begin{vmatrix} f_u & f_w \\ g_u & g_w \end{vmatrix}.$$

PROBLEMS

1. Obtain the Jacobian matrix for each of the following mappings:

a) $y_1 = 5x_1 + 2x_2$, $y_2 = 2x_1 + 3x_2$.

b) $y_1 = 2x_1^2 + x_2^2$, $y_2 = 3x_1x_2$.

c) $y_1 = x_1x_2x_3$, $y_2 = x_1^2x_3$.

d) $u = x\cos y$, $v = x\sin y$, $w = x^2$.

e) $w = x^2yz$.

f) $w = x^2 + y^2 - z^2$.

g) $x = t^2$, $y = t^3$, $z = t^4$.

2. Obtain the linear mapping $d\mathbf{y} = \mathbf{f_x}\, d\mathbf{x}$ approximating the given mapping $\mathbf{y} = \mathbf{f}(\mathbf{x})$ near the specified point and use the linear mapping to obtain an approximation to the value $\mathbf{f}(\mathbf{x})$ specified.

a) $y_1 = x_1^2 + x_2^2$, $y_2 = x_1x_2$ at $(2, 1)$, approximate $\mathbf{f}(2.04, 1.01)$.

b) $y_1 = x_1x_2 - x_3^2$, $y_2 = x_1x_2 + x_1x_3$ at $(3, 2, 1)$, approximate $\mathbf{f}(3.01, 1.99, 1.03)$.

c) $u = e^x\cos y$, $v = e^x\sin y$, $w = 2e^x$ at $(0, \pi/2)$, approximate value of (u, v, w) for $(x, y) = (0.1, 1.6)$.

d) $y_1 = x_2^2 + \cdots + x_n^2$, $y_2 = x_1^2 + x_3^2 + \cdots + x_n^2, \ldots, y_n = x_1^2 + \cdots + x_{n-1}^2$ at $(1, 0, \ldots, 0)$, approximate $\mathbf{f}(1, 0.1, \ldots, 0.1)$.

3. Obtain the Jacobian determinant requested:

a) $\frac{\partial(u, v)}{\partial(x, y)}$ for $u = x^3 - 3xy^2$, $v = 3x^2y - y^3$.

b) $\frac{\partial(u, v, w)}{\partial(x, y, z)}$ for $u = xe^y\cos z$, $v = xe^y\sin z$, $w = xe^y$.

c) $\frac{\partial(f, g)}{\partial(u, v)}$ for $f(u, v, w) = u^2vw$, $g(u, v, w) = u^2v^2 - w^4$.

d) $\frac{\partial(f,g,h)}{\partial(x,y,z)}$ for $f(x,y,z,t) = x^2+2y+z^2-t^2$, $g(x,y,z,t) = xyz+t^2$, $h(x,y,z,t) = z^2-t^2$.

4. For the mapping $u = e^x \cos y$, $v = e^x \sin y$ from the xy-plane to the uv-plane, carry out the following steps:

a) Evaluate the Jacobian determinant at $(1, 0)$.

b) Show that the square R_{xy}: $0.9 \le x \le 1.1$, $-0.1 \le y \le 0.1$ corresponds to the region R_{uv} bounded by arcs of the circles $u^2+v^2 = e^{1.8}$, $u^2+v^2 = e^{2.2}$ and the rays $v = \pm(\tan 0.1)\,u$, $u \ge 0$, and find the ratio of the area of R_{uv} to that of R_{xy}. Compare with the result of (a).

c) Obtain the approximating linear mapping at $(1, 0)$ and find the region R'_{uv} corresponding to the square R_{xy} of part (b) under this linear mapping. Find the ratio of the area of R'_{uv} to that of R_{xy} and compare with the results of parts (a) and (b).

5. a) Let $\mathbf{u}$, $\mathbf{v}$ be linearly independent vectors in V^2. Show by geometric reasoning that the points P of the plane for which

$$\mathbf{x} = \overrightarrow{OP} = a\mathbf{u} + b\mathbf{v}, \quad 0 \le a \le 1,\ 0 \le b \le 1,$$

fill a parallelogram whose edges, properly directed, represent $\mathbf{u}$ and $\mathbf{v}$.

b) With $\mathbf{u}$, $\mathbf{v}$ as in (a), let A be a nonsingular 2×2 matrix, so that $A\mathbf{u}$, $A\mathbf{v}$ are also linearly independent (Problem 13 following Section 1.16) and under the linear mapping $\mathbf{y} = A\mathbf{x}$ the parallelogram of part (a) is mapped onto a parallelogram given by

$$\mathbf{y} = \overrightarrow{OQ} = A(a\mathbf{u} + b\mathbf{v}) = aA\mathbf{u} + bA\mathbf{v}, \quad 0 \le a \le 1,\ 0 \le b \le 1,$$

in the plane.
Show that the area of the second parallelogram is $|\det A|$ times the area of the first. [Hint: Let B be the matrix whose column vectors are $\mathbf{u}$, $\mathbf{v}$ and let C be the matrix whose column vectors are $A\mathbf{u}$, $A\mathbf{v}$. Show that the areas in question are $|\det B|$ and $|\det C|$.]

6. Generalize the results of Problem 5 to 3-dimensional space. Thus in (a) use three vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ and consider the corresponding parallelepiped and in (b) take A to be a nonsingular 3×3 matrix and consider volumes.

2.8 Derivatives and Differentials of Composite Functions

The functions to be considered in the following will be assumed to be defined in appropriate domains and to have continuous first partial derivatives, so that the corresponding differentials can be formed.

THEOREM If $z = f(x, y)$ and $x = g(t)$, $y = h(t)$, then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}. \tag{2.33}$$

If $z = f(x, y)$ and $x = g(u, v)$, $y = h(u, v)$, then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}, \qquad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}. \tag{2.34}$$