Finally,

$$\frac{\partial G}{\partial w} = \frac{\partial}{\partial w}\int_v^u f(x, w)\,dx = \int_v^u \frac{\partial f}{\partial w}(x, w)\,dx$$

by Leibnitz's Rule. Since $w = t$, $dw/dt = 1$ and the third term is accounted for. ●

PROBLEMS

1. Obtain the indicated derivatives in the form of integrals:

a) $\frac{d}{dt}\int_{\frac{\pi}{2}}^{\pi} \frac{\cos(xt)}{x}\,dx$ **b)** $\frac{d}{dt}\int_1^2 \frac{x^2}{(1-tx)^2}\,dx$

c) $\frac{d}{du}\int_1^2 \log(xu)\,dx$ **d)** $\frac{d^n}{dy^n}\int_1^2 \frac{\sin x}{x-y}\,dx$

2. Obtain the indicated derivatives:

a) $\frac{d}{dx}\int_1^x t^2 dt$ **b)** $\frac{d}{dt}\int_1^{t^2} \sin(x^2)\,dx$

c) $\frac{d}{dt}\int_{t^3}^2 \log(1+x^2)\,dx$ **d)** $\frac{d}{dx}\int_x^{\tan x} e^{-t^2}\,dt$

3. Prove the following:

a) $\frac{d}{d\alpha}\int_{\sin\alpha}^{\cos\alpha} \log(x+\alpha)\,dx = \log\frac{\cos\alpha+\alpha}{\sin\alpha+\alpha} - [\sin\alpha\log(\cos\alpha+\alpha) + \cos\alpha\log(\sin\alpha+\alpha)]$;

b) $\frac{d}{du}\int_0^{\frac{\pi}{2u}} u\sin ux\,dx = 0$;

c) $\frac{d}{dy}\int_y^{y^2} e^{-x^2y^2}\,dx = 2ye^{-y^6} - e^{-y^4} - 2y\int_y^{y^2} x^2e^{-x^2y^2}\,dx$.

4. a) Evaluate $\int_0^1 x^n \log x\,dx$ by differentiating both sides of the equation $\int_0^1 x^n\,dx = \frac{1}{n+1}$ with respect to n ($n > -1$).

b) Evaluate $\int_0^\infty x^n e^{-ax}\,dx$ by repeated differentiation of $\int_0^\infty e^{-ax}\,dx$ ($a > 0$).

c) Evaluate $\int_0^\infty \frac{dy}{(x^2+y^2)^n}$ by repeated differentiation of $\int_0^\infty \frac{dy}{x^2+y^2}$.

[In (b) and (c) the improper integrals are of a type to which Leibnitz's Rule is applicable, as is shown in Chapter 6. The result of (a) can be explicitly verified.]

5. Leibnitz's Rule extends to indefinite integrals in the form:

$$\frac{\partial}{\partial t}\int f(x, t)\,dx + C = \int \frac{\partial}{\partial t} f(x, t)\,dx. \quad \text{(a)}$$

There is still an arbitrary constant in the equation because we are evaluating an *indefinite* integral. Thus from the equation

$$\int e^{tx}\,dx = \frac{e^{tx}}{t} + C,$$

one deduces that

$$\int xe^{tx}\,dx = e^{tx}\left(\frac{x}{t} - \frac{1}{t^2}\right) + C_1.$$

a) By differentiating n times, prove that

$$\int \frac{dx}{(x^2+a)^n} = \frac{(-1)^{n-1}}{(n-1)!}\frac{\partial^{n-1}}{\partial a^{n-1}}\left(\frac{1}{\sqrt{a}}\arctan\frac{x}{\sqrt{a}}\right) + C \quad (a > 0).$$

b) Prove $\int x^n \cos ax\,dx = \frac{\partial^n}{\partial a^n}\left(\frac{\sin ax}{a}\right) + C, n = 4, 8, 12, \ldots$.

c) Let $\int f(x,t)\,dx = F(x,t) + C$, so that $\partial F/\partial x = f(x,t)$. Show that Eq. (a) is equivalent to the statement

$$\frac{\partial^2 F}{\partial x\,\partial t} = \frac{\partial^2 F}{\partial t\,\partial x}.$$

6. It is known that

$$\int_0^{2\pi} \frac{\cos\theta}{1 - a\cos\theta}\,d\theta = 2\pi\,\frac{1-\sqrt{1-a^2}}{a\sqrt{1-a^2}},$$

where a is a constant, $0 < a < 1$. (This can be established as in elementary calculus with the aid of the substitution $t = \tan(\theta/2)$.) Use this result to prove that

$$\int_0^{2\pi} \log(1 - a\cos\theta)\,d\theta = 2\pi \log\frac{1+\sqrt{1-a^2}}{2}.$$

[Hint: Call the left-hand side of the new equation $g(a)$, find $g'(a)$, and integrate to find $g(a) = 2\pi\log(1+\sqrt{1-a^2}) + C$. Use continuity of g for $a = 0$ and $g(0) = 0$ to find C.]

7. Consider a 1-dimensional fluid motion, the flow taking place along the x axis. Let $v = v(x,t)$ be the velocity at position x at time t, so that if x is the coordinate of a fluid particle at time t, one has $dx/dt = v$. If $f(x,t)$ is any scalar associated with the flow (velocity, acceleration, density, ...), one can study the variation of f following the flow with the aid of the Stokes derivative:

$$\frac{Df}{Dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial t}$$

[see Problem 12 following Section 2.8]. A piece of the fluid occupying an interval $a_0 \le x \le b_0$ when $t = 0$ will occupy an interval $a(t) \le x \le b(t)$ at time t, where $\frac{da}{dt} = v(a,t)$, $\frac{db}{dt} = v(b,t)$. The integral

$$F(t) = \int_{a(t)}^{b(t)} f(x,t)\,dx$$

is then an integral of f over a definite piece of the fluid, whose position varies with time; if f is density, this is the mass of the piece. Show that

$$\frac{dF}{dt} = \int_{a(t)}^{b(t)} \left[\frac{\partial f}{\partial t}(x,t) + \frac{\partial}{\partial x}(fv)\right] dx = \int_{a(t)}^{b(t)} \left(\frac{Df}{Dt} + f\frac{dv}{dx}\right) dx.$$

This is generalized to arbitrary 3-dimensional flows in Section 5.15.

8. Let $f(\alpha)$ be continuous for $0 \le \alpha \le 2\pi$. Let

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha)\frac{1-r^2}{1+r^2-2r\cos(\theta-\alpha)}\,d\alpha$$

for $r < 1$, r and θ being polar coordinates. Show that u is harmonic for $r < 1$. This is the *Poisson integral formula*. [Hint: Write $w = 1 + r^2 - 2r\cos(\theta - \alpha)$ and $v(r,\theta,\alpha) = (1-r^2)w^{-1}$. Use Leibnitz's Rule to conclude that $\nabla^2 u = (2\pi)^{-1}\int_0^{2\pi} f(\alpha)\nabla^2 v\,d\theta$, where $\nabla^2 v = v_{rr} + r^{-2}v_{\theta\theta} + r^{-1}v_r$ as in Eq. (2.138) in Section 2.17. Show that $v_r = -2rw^{-1} - (1-r^2)w^{-2}w_r$ etc. and finally

$$\nabla^2 v = -4w^{-1} + (5r - r^{-1})ww_r + (r^2-1)\big(w^{-2}w_{rr} - 2w^{-3}w_r^2$$
$$+ r^{-2}w^{-2}w_{\theta\theta} - 2r^{-2}w^{-3}w_\theta^2\big).$$

Multiply both sides by r^2w^3, insert the proper expressions for $w, w_r, \ldots$ on the right and collect terms in powers of r ($r^6, r^5, \ldots$) to verify that $r^2w^3\nabla^2 v = 0$ and hence $\nabla^2 u = 0$.]