A second type of improper integral, generalizing the definite integrals with infinite limits, is an integral

$$\iint_R f(x, y)\, dx\, dy$$

where R is an *unbounded* closed region. Here one obtains a value by a limit process just like that of (4.91). The most important case of this is that of a function continuous outside and on a circle $x^2 + y^2 = a^2$. If $f(x, y)$ is of one sign, the integral over this region R can be defined as the limit

$$\lim_{k \to \infty} \iint_{R_k} f(x, y)\, dx\, dy$$

where R_k is the region $a^2 \leq x^2 + y^2 \leq k^2$. Thus the improper integral

$$\iint_R \frac{1}{r^p}\, dx\, dy$$

has the value

$$\lim_{k \to \infty} \int_a^k \int_0^{2\pi} \frac{1}{r^p}\, d\theta\, r\, dr = \lim_{k \to \infty} 2\pi \frac{k^{2-p} - a^{2-p}}{2 - p},$$

which equals $2\pi a^{2-p}/(p - 2)$, for $p > 2$. For $p \leq 2$ the integral diverges.

While the emphasis here has been on double integrals, the statements hold with minor changes [affecting in particular the critical value of p for the integral (4.92)] for triple and other multiple integrals.

For further discussion of this topic, see Section 6.26.

PROBLEMS

1. One way of evaluating the *error integral*

$$\int_0^\infty e^{-x^2}\, dx$$

is to use the equations

$$\left(\int_0^\infty e^{-x^2}\, dx\right)^2 = \int_0^\infty e^{-x^2}\, dx \int_0^\infty e^{-y^2}\, dy = \int_0^\infty \int_0^\infty e^{-x^2-y^2}\, dx\, dy$$

and to evaluate the double integral by polar coordinates. Carry out this evaluation, showing that the integral equals $\frac{1}{2}\sqrt{\pi}$; also discuss the significance of the above equations in terms of the limit definitions of the improper integrals.

2. Show that the integral

$$\iint_R \log \sqrt{x^2 + y^2}\, dx\, dy$$

converges, where R is the region $x^2 + y^2 \leq 1$, and find its value. This can be interpreted as minus the *logarithmic potential*, at the origin, of a uniform mass distribution over the circle.

3. a) Show that the integral

$$\iiint_R \frac{1}{r^p}\,dx\,dy\,dz, \quad r = \sqrt{x^2+y^2+z^2},$$

over the spherical region $x^2+y^2+z^2 \le 1$ converges for $p < 3$ and find its value. For $p = 1$ this is the *Newtonian potential* of a uniform mass distribution over the solid sphere, evaluated at the origin.

b) For the integral of part (a), let R be the *exterior* region $x^2+y^2+z^2 \ge 1$. Show that the integral converges for $p > 3$ and find its value.

4. Test for convergence or divergence:

a) $\iint_R \frac{1}{x^2+y^2}\,dx\,dy$, over the square $|x| < 1$, $|y| < 1$;

b) $\iint_R \frac{\log(x^2+y^2)}{\sqrt{x^2+y^2}}\,dx\,dy$ over the circle $x^2+y^2 \le 1$;

c) $\iint_R \log(x^2+y^2)\,dx\,dy$ over the region $x^2+y^2 \ge 1$;

d) $\iint_R \frac{\sqrt{x^2+xy+y^2}}{x^2+y^2}\,dx\,dy$ over the region $x^2+y^2 \le 1$;

e) $\iiint_R \log(x^2+y^2+z^2)\,dx\,dy\,dz$ over the solid $x^2+y^2+z^2 \le 1$.

4.9 Integrals Depending on a Parameter ■ Leibnitz's Rule

A definite integral

$$\int_a^b f(x,t)\,dx$$

of a continuous function $f(x,t)$ has a value that depends on the choice of t, so that one can write

$$\int_a^b f(x,t)\,dx = F(t). \tag{4.93}$$

One calls such an expression an *integral depending on a parameter*, and t is termed the parameter. Thus

$$\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1-k^2\sin^2 x}}$$

is an integral depending on the parameter k; this example happens to be a *complete elliptic integral* (Section 4.2).

If an integral depending on a parameter can be evaluated in terms of familiar functions, it becomes simply an explicit function of one variable. Thus, for example,

$$\int_0^{\pi} \sin(xt)\,dx = \frac{1}{t} - \frac{\cos(\pi t)}{t} \quad (t \ne 0).$$

However, it can easily happen, as the preceding elliptic integral illustrates, that the integral cannot be expressed in terms of familiar functions. In such a case the function of the parameter is nevertheless well defined. It can be evaluated as accurately as desired for each particular parameter value and then tabulated; precisely this has