8. Determine the solution of each of the following problems:

(a) $\nabla^2 u = 0, \qquad 1 < r < 2, \qquad 0 < \theta < \pi,$

$u(1,\theta) = \sin\theta, \qquad u(2,\theta) = 0, \qquad 0 \le \theta \le \pi,$

$u(r,0) = 0, \qquad u(r,\pi) = 0, \qquad 1 \le r \le 2.$

(b) $\nabla^2 u = 0, \qquad 1 < r < 2, \qquad 0 < \theta < \pi,$

$u(1,\theta) = 0, \qquad u(2,\theta) = \theta(\theta - \pi), \quad 0 \le \theta \le \pi,$

$u(r,0) = 0, \qquad u(r,\pi) = 0, \qquad 1 \le r \le 2.$

(c) $\nabla^2 u = 0, \qquad 1 < r < 3, \qquad 0 < \theta < \pi/2,$

$u(1,\theta) = 0, \qquad u(3,\theta) = 0, \qquad 0 \le \theta \le \pi/2,$

$u(r,0) = (r-1)(r-3), \quad u\left(r, \frac{\pi}{2}\right) = 0, \qquad 1 \le r \le 3.$

(d) $\nabla^2 u = 0, \qquad 1 < r < 3, \qquad 0 < \theta < \pi/2,$

$u(1,\theta) = 0, \qquad u(3,\theta) = 0, \qquad 0 \le \theta \le \pi,$

$u(r,0) = 0, \qquad u\left(r, \frac{\pi}{2}\right) = f(r), \qquad 1 \le r \le 3.$

9. Solve the boundary-value problem

$$\nabla^2 u = 0, \qquad a < r < b, \qquad 0 < \theta < \alpha,$$

$$u(a,\theta) = f(\theta), \quad u(b,\theta) = 0, \qquad 0 \le \theta \le \alpha,$$

$$u(r,\alpha) = 0, \qquad u(r,0) = f(r), \quad a \le r \le b.$$

10. Verify directly that the Poisson integral is a solution of the Laplace equation.

11. Solve

$$\nabla^2 u = 0, \qquad 0 < r < a, \qquad 0 < \theta < \pi,$$

$$u(r,0) = 0, \qquad u(r,\pi) = 0,$$

$$u(a,\theta) = \theta(\pi - \theta), \quad 0 \le \theta \le \pi,$$

$$u(0,\theta) \text{ is bounded.}$$