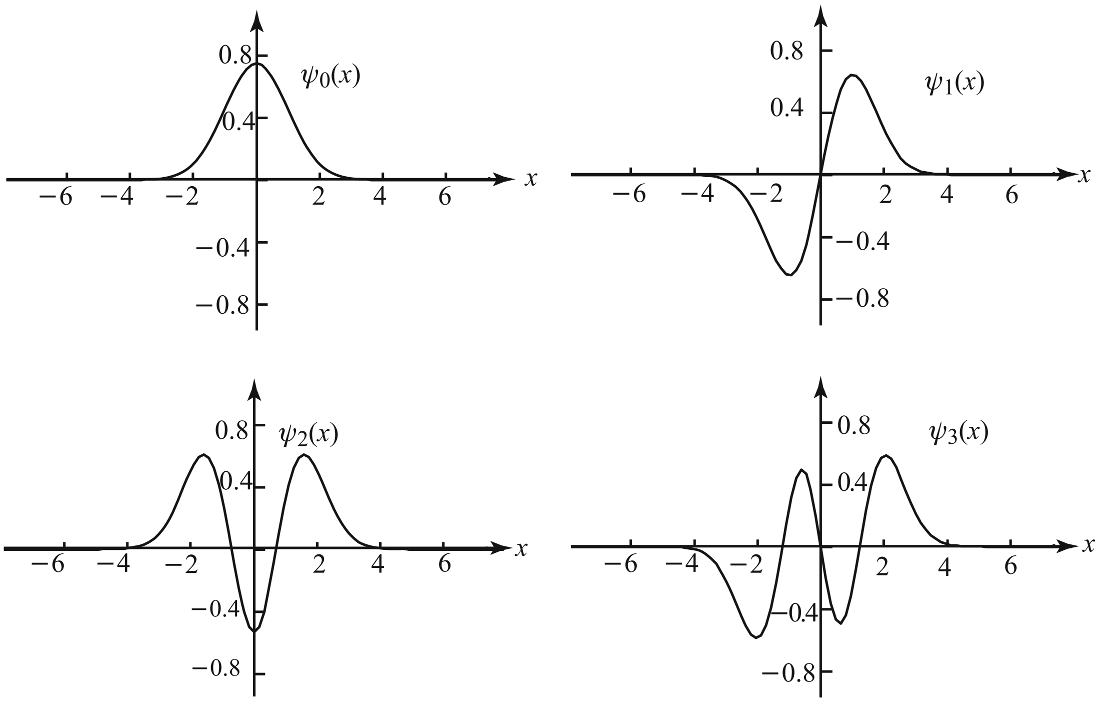


**Figure 8.13.1** Eigenfunctions $\psi_n$ for $n = 0, 1, 2, 3$.

## 8.14 Exercises

1. Determine the eigenvalues and eigenfunctions of the following regular Sturm–Liouville systems:

   (a) $y'' + \lambda y = 0,$

   $$y(0) = 0, \; y(\pi) = 0.$$

   (b) $y'' + \lambda y = 0,$

   $$y(0) = 0, \; y'(1) = 0.$$

   (c) $y'' + \lambda y = 0,$

   $$y'(0) = 0, \; y'(\pi) = 0.$$

   (d) $y'' + \lambda y = 0,$

   $$y(1) = 0, \; y(0) + y'(0) = 0.$$

2. Find the eigenvalues and eigenfunctions of the following periodic Sturm–Liouville systems:

   (a) $y'' + \lambda y = 0,$

   $$y\,(-1) = y\,(-1)\,,\ y'\,(-1) = y'\,(1).$$

   (b) $y'' + \lambda y = 0,$

   $$y\,(0) = y\,(2\pi)\,,\ y'\,(0) = y'\,(2\pi).$$

   (c) $y'' + \lambda y = 0,$

   $$y\,(0) = y\,(\pi)\,,\ y'\,(0) = y'\,(\pi).$$

3. Obtain the eigenvalues and eigenfunctions of the following Sturm–Liouville systems:

   (a) $y'' + y' + (1+\lambda)\,y = 0,$

   $$y\,(0) = 0,\ y\,(1) = 0.$$

   (b) $y'' + 2y' + (1-\lambda)\,y = 0,$

   $$y\,(0) = 0,\ y'\,(1) = 0.$$

   (c) $y'' - 3y' + 3\,(1+\lambda)\,y = 0,$

   $$y'\,(0) = 0,\ y'\,(\pi) = 0.$$

4. Find the eigenvalues and eigenfunctions of the following regular Sturm–Liouville systems:

   (a) $x^2 y'' + 3xy' + \lambda y = 0,\ 1 \le x \le e,$

   $$y\,(1) = 0,\ y\,(e) = 0.$$

   (b) $\frac{d}{dx}\left[(2+x)^2\,y'\right] + \lambda y = 0,\ -1 \le x \le 1,$

   $$y\,(-1) = 0,\ y\,(1) = 0.$$

   (c) $(1+x)^2\,y'' + 2\,(1+x)\,y' + 3\lambda y = 0,\ 0 \le x \le 1,$

   $$y\,(0) = 0,\ y\,(1) = 0.$$

5. Determine all eigenvalues and eigenfunctions of the Sturm–Liouville systems:

   (a) $x^2 y'' + xy' + \lambda y = 0,$

   $$y(1) = 0,\ y, y' \text{ are bounded at } x = 0.$$

   (b) $y'' + \lambda y = 0,$

   $$y(0) = 0,\ y, y' \text{ are bounded at infinity.}$$

6. Expand the function

$$f(x) = \sin x, \quad 0 \le x \le \pi$$

   in terms of the eigenfunctions of the Sturm–Liouville problem

$$y'' + \lambda y = 0,$$
$$y(0) = 0,\ y(\pi) + y'(\pi) = 0.$$

7. Find the expansion of

$$f(x) = x, \qquad 0 \le x \le \pi$$

   in a series of eigenfunctions of the Sturm–Liouville system

$$y'' + \lambda y = 0,$$
$$y'(0) = 0,\ y'(\pi) = 0.$$

8. Transform each of the following equations into the equivalent self-adjoint form:
   (a) The Laguerre equation

$$xy'' + (1 - x) y' + ny = 0, \qquad n = 0, 1, 2, \ldots.$$

   (b) The Hermite equation

$$y'' - 2xy' + 2ny = 0, \qquad n = 0, 1, 2, \ldots.$$

   (c) The Tchebycheff equation

$$\left(1 - x^2\right) y'' - xy' + n^2 y = 0, \qquad n = 0, 1, 2, \ldots.$$

9. If $q(x)$ and $s(x)$ are continuous and $p(x)$ is twice continuously differentiable in $[a, b]$, show that the solutions of the fourth-order Sturm–Liouville system