We have, from (6.13.12),

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos kx\, dk \int_0^1 \cos kt\, dt = \frac{2}{\pi}\int_0^\infty \left(\frac{\sin k}{k}\right)\cos kx\, dk,$$

or,

$$1 = \frac{2}{\pi}\int_0^\infty \frac{\sin k}{k}\cos kx\, dk.$$

6.14 Exercises

1. Find the Fourier series of the following functions:

   (a) $f(x) = \begin{cases} x & -\pi < x < 0 \\ h & 0 < x < \pi, \end{cases}$ h is a constant

   (b) $f(x) = \begin{cases} 1 & -\pi < x < 0 \\ x^2 & 0 < x < \pi, \end{cases}$

   (c) $f(x) = x + \sin x \quad -\pi < x < \pi,$

   (d) $f(x) = 1 + x \quad -\pi < x < \pi,$

   (e) $f(x) = e^x \quad -\pi < x < \pi,$

   (f) $f(x) = 1 + x + x^2 \quad -\pi < x < \pi.$

2. Determine the Fourier sine series of the following functions:

   (a) $f(x) = \pi - x \quad 0 < x < \pi,$

   (b) $f(x) = \begin{cases} 1 & 0 < x < \pi/2 \\ 2 & \pi/2 < x < \pi, \end{cases}$

   (c) $f(x) = x^2 \quad 0 < x < \pi,$

   (d) $f(x) = \cos x \quad 0 < x < \pi,$

   (e) $f(x) = x^3 \quad 0 < x < \pi,$

   (f) $f(x) = e^x \quad 0 < x < \pi.$

3. Obtain the Fourier cosine series representation for the following functions:

   (a) $f(x) = \pi + x \qquad 0 < x < \pi,$

   (b) $f(x) = x \qquad 0 < x < \pi,$

   (c) $f(x) = x^2 \qquad 0 < x < \pi,$

   (d) $f(x) = \sin 3x \qquad 0 < x < \pi,$

   (e) $f(x) = e^x \qquad 0 < x < \pi,$

   (f) $f(x) = \cosh x \qquad 0 < x < \pi.$

4. Expand the following functions in a Fourier series:

   (a) $f(x) = x^2 + x \qquad -1 < x < 1,$

   (b) $f(x) = \begin{cases} 1 & 0 < x < 3 \\ 0 & 3 < x < 6, \end{cases}$

   (c) $f(x) = \sin(\pi x/l) \qquad 0 < x < l,$

   (d) $f(x) = x^3 \qquad -2 < x < 2,$

   (e) $f(x) = e^{-x} \qquad 0 < x < 1,$

   (f) $f(x) = \sinh x \qquad -1 < x < 1.$

5. Expand the following functions in a complex Fourier series:

   (a) $f(x) = e^{2x} \qquad -\pi < x < \pi,$

   (b) $f(x) = \cosh x \qquad -\pi < x < \pi,$

   (c) $f(x) = \begin{cases} 1 & -\pi < x < 0 \\ \cos x & 0 < x < \pi, \end{cases}$

   (d) $f(x) = x \qquad -1 < x < 1,$

   (e) $f(x) = x^2 \qquad -\pi < x < \pi,$

   (f) $f(x) = \sinh(\pi x/2) \qquad -2 < x < 2.$

6. (a) Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x/2, & 0 < x < \pi. \end{cases}$$

(b) With the use of the Fourier series of $f(x)$ in 6(a), show that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots.$$

7. (a) Determine the Fourier series of the function

$$f(x) = x^2, \qquad -l < x < l.$$

(b) With the use of the Fourier series of $f(x)$ in 7(a), show that

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots.$$

8. Determine the Fourier series expansion of each of the following functions by performing the differentiation of the appropriate Fourier series:

(a) $\sin^2 x$ $\qquad 0 < x < \pi,$

(b) $\cos^2 x$ $\qquad 0 < x < \pi,$

(c) $\sin x \cos x$ $\qquad 0 < x < \pi,$

(d) $\cos x + \cos 2x$ $\qquad 0 < x < \pi,$

(e) $\cos x + \cos 2x$ $\qquad 0 < x < \pi.$

9. Find the functions represented by the new series which are obtained by termwise integration of the following series from 0 to x:

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin kx = x/2 \qquad -\pi < x < \pi,$

(b) $\displaystyle\frac{3}{2} + \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{1-(-1)^k}{k} \sin kx = \begin{cases} 1 & -\pi < x < 0 \\ 2 & 0 < x < \pi, \end{cases}$

(c) $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{\cos kx}{k} = \ln\left(2\cos\frac{x}{2}\right) \qquad -\pi < x < \pi,$

(d) $\sum_{k=1}^{\infty} \frac{\sin(2k+1)x}{(2k+1)^3} = \frac{\pi^2 x - \pi x^2}{8}$ $\qquad 0 < x < 2\pi,$

(e) $\left(\frac{4}{\pi}\right) \sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{(2k-1)} = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi. \end{cases}$

10. Determine the double Fourier series of the following functions:

(a) $f(x,y) = 1$ $\qquad 0 < x < \pi \qquad 0 < y < \pi,$

(b) $f(x,y) = xy^2$ $\qquad 0 < x < \pi \qquad 0 < y < \pi,$

(c) $f(x,y) = x^2y^2$ $\qquad 0 < x < \pi \qquad 0 < y < \pi,$

(d) $f(x,y) = x^2 + y$ $\qquad -\pi < x < \pi \qquad -\pi < y < \pi,$

(e) $f(x,y) = x \sin y$ $\qquad -\pi < x < \pi \qquad -\pi < y < \pi,$

(f) $f(x,y) = e^{x+y}$ $\qquad -\pi < x < \pi \qquad -\pi < y < \pi,$

(g) $f(x,y) = xy$ $\qquad 0 < x < 1 \qquad 0 < y < 2,$

(h) $f(x,y) = 1$ $\qquad 0 < x < a \qquad 0 < y < b,$

(i) $f(x,y) = x \cos y$ $\qquad -1 < x < 1 \qquad -2 < y < 2,$

(j) $f(x,y) = xy^2$ $\qquad -\pi < x < \pi \qquad -\pi < y < \pi,$

(k) $f(x,y) = x^2y^2$ $\qquad -\pi < x < \pi \qquad -\pi < y < \pi.$

11. Deduce the general double Fourier series expansion formula for the function $f(x,y)$ in the rectangle $-a < x < a$, $-b < y < b$.

12. Prove the *Weierstrass Approximation Theorem*: If f is a continuous function on the interval $-\pi \le x \le \pi$ and if $f(-\pi) = f(\pi)$, then, for any $\varepsilon > 0$, there exists a trigonometric polynomial

$$T(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

such that

$$|f(x) - T(x)| < \varepsilon$$

for all x in $[-\pi, \pi]$.