A general solution to the nonhomogeneous equation (1) can be written as

$$y(x) = y_p(x) + y_h(x) ,$$

where y_p is some particular solution to (1) and y_h is a general solution to the corresponding homogeneous equation. Two useful techniques for finding particular solutions are the **annihilator method** (undetermined coefficients) and the method of **variation of parameters.**

The annihilator method applies to equations of the form

$$L[y] = g(x) , \tag{6}$$

where L is a linear differential operator with constant coefficients and the forcing term $g(x)$ is a polynomial, exponential, sine, or cosine, or a linear combination of products of these. Such a function $g(x)$ is annihilated (mapped to zero) by a linear differential operator A that also has constant coefficients. Every solution to the nonhomogeneous equation (6) is then a solution to the homogeneous equation $AL[y] = 0$, and, by comparing the solutions of the latter equation with a general solution to $L[y] = 0$, we can obtain the *form* of a particular solution to (6). These forms have previously been studied in Section 4.4 for the method of undetermined coefficients.

The method of variation of parameters is more general in that it applies to arbitrary equations of the form (1). The idea is, starting with a fundamental solution set $\{y_1, \dots, y_n\}$ for (2), to determine functions $v_1, \dots, v_n$ such that

$$y_p(x) = v_1(x)y_1(x) + \cdots + v_n(x)y_n(x) \tag{7}$$

satisfies (1). This method leads to the formula

$$y_p(x) = \sum_{k=1}^{n} y_k(x) \int \frac{g(x)W_k(x)}{W[y_1, \dots, y_n](x)}\, dx , \tag{8}$$

where

$$W_k(x) = (-1)^{n-k}W[y_1, \dots, y_{k-1}, y_{k+1}, \dots, y_n](x) , \qquad k = 1, \dots, n .$$

REVIEW PROBLEMS

1. Determine the intervals for which Theorem 1 on page 343 guarantees the existence of a solution in that interval.
   (a) $y^{(4)} - (\ln x)y'' + xy' + 2y = \cos 3x$
   (b) $(x^2 - 1)y''' + (\sin x)y'' + \sqrt{x + 4}\, y' + e^x y = x^2 + 3$
2. Determine whether the given functions are linearly dependent or linearly independent on the interval $(0, \infty)$.
   (a) $\{e^{2x}, x^2e^{2x}, e^{-x}\}$
   (b) $\{e^x \sin 2x, xe^x \sin 2x, e^x, xe^x\}$
   (c) $\{2e^{2x} - e^x, e^{2x} + 1, e^{2x} - 3, e^x + 1\}$
3. Show that the set of functions $\{\sin x, x \sin x, x^2 \sin x, x^3 \sin x\}$ is linearly independent on $(-\infty, \infty)$.
4. Find a general solution for the given differential equation.
   (a) $y^{(4)} + 2y''' - 4y'' - 2y' + 3y = 0$
   (b) $y''' + 3y'' - 5y' + y = 0$
   (c) $y^{(5)} - y^{(4)} + 2y''' - 2y'' + y' - y = 0$
   (d) $y''' - 2y'' - y' + 2y = e^x + x$

5. Find a general solution for the homogeneous linear differential equation with constant coefficients whose auxiliary equation is

(a) $(r+5)^2(r-2)^3(r^2+1)^2 = 0$.

(b) $r^4(r-1)^2(r^2+2r+4)^2 = 0$.

6. Given that $y_p = \sin(x^2)$ is a particular solution to

$$y^{(4)} + y = (16x^4 - 11)\sin(x^2) - 48x^2\cos(x^2)$$

on $(0, \infty)$, find a general solution.

7. Find a differential operator that annihilates the given function.

(a) $x^2 - 2x + 5$ **(b)** $e^{3x} + x - 1$

(c) $x \sin 2x$ **(d)** $x^2e^{-2x}\cos 3x$

(e) $x^2 - 2x + xe^{-x} + \sin 2x - \cos 3x$

8. Use the annihilator method to determine the form of a particular solution for the given equation.

(a) $y'' + 6y' + 5y = e^{-x} + x^2 - 1$

(b) $y''' + 2y'' - 19y' - 20y = xe^{-x}$

(c) $y^{(4)} + 6y'' + 9y = x^2 - \sin 3x$

(d) $y''' - y'' + 2y = x \sin x$

9. Find a general solution to the Cauchy–Euler equation

$$x^3y''' - 2x^2y'' - 5xy' + 5y = x^{-2} ,$$
$$x > 0 ,$$

given that $\{x, x^5, x^{-1}\}$ is a fundamental solution set to the corresponding homogeneous equation.

10. Find a general solution to the given Cauchy–Euler equation.

(a) $4x^3y''' + 8x^2y'' - xy' + y = 0 , \quad x > 0$

(b) $x^3y''' + 2x^2y'' + 2xy' + 4y = 0 , \quad x > 0$

TECHNICAL WRITING EXERCISES

1. Describe the differences and similarities between second-order and higher-order linear differential equations. Include in your comparisons both theoretical results and the methods of solution. For example, what complications arise in solving higher-order equations that are not present for the second-order case?

2. Explain the relationship between the method of undetermined coefficients and the annihilator method. What difficulties would you encounter in applying the annihilator method if the linear equation did not have constant coefficients?

3. For students with a background in linear algebra: Compare the theory for kth-order linear differential equations with that for systems of n linear equations in n unknowns whose coefficient matrix has rank $n - k$. Use the terminology from linear algebra; for example, subspaces, basis, dimension, linear transformation, and kernel. Discuss both homogeneous and nonhomogeneous equations.